A MODEL FOR THE PEANO SURFACE.

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1. Introduction. In his *Calcolo differentiale e principi di calcolo integrale*, published in 1884, Peano gave the first rigorous treatment of the theory of maxima and minima of functions of several variables. The development of this theory may also be found on pp. 181–188, in notes 133–136, and on p. 332 of the German translation (1899). The note contains a discussion of the now famous function, representing a certain quartic surface which may properly be called the Peano surface.[1] By this example Peano demonstrated the falsity of some of the hitherto accepted criteria for maxima and minima of functions of several variables. One of these false criteria will be mentioned in the constructive discussion of the Peano surface. The investigations of Peano induced others to penetrate still more deeply into the theory of maxima and minima[2] and related domains of the calculus of variations. Some very reputable mathematicians and authors, like Serret,[3] Bertrand,[4] Todhunter,[5] and others, have fallen into this pit of faulty criteria.

The first one in America to call attention to these defects was Professor J. Pierpont in an interesting article on maxima and minima of functions of several variables.[6]

In most cases the errors are fundamentally due to the theorem, which in general is not true, that in Taylor's development of a function of several variables, the ratio of the remainder, following a certain term of the expansion, to this term, approaches zero as a limit when the increments of the variables approach zero as a limit. For cases in which this limit is zero, see Peano (German translation, theorem III, p. 173).

It is the purpose of this paper to discuss the geometrical properties of the Peano surface and to make them intuitively apparent by the aid of a model constructed by the writer. Peano's important criticisms can be comprehended very readily by means of this geometric visualization.

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[1] Georg Scheffers, *Lehrbuch der darstellenden Geometrie*, vol. 2, 1920, pp. 261–263.

[2] L. Scheffer, "Theorie der Maxima und Minima einer Funktion von zwei Variablen," *Mathematische Annalen*, vol. 35, 1890, pp. 541–576. Scheffer arrived independently at some of the conclusions of Peano, but uses Peano's function as an example.

V. v. Dantscher, "Zur Theorie der Maxima und Minima einer Funktion von zwei Veränderlichen," *ibid.*, vol. 42, 1893, pp. 89–131.

O. Stolz, *Grundzüge der Differential- und Integralrechnung*, vol. 1, Leipzig, 1893, pp. 213–228. Also *Sitzungsberichte der math.-natur. Klasse der Akademie des Wissenschaften*, Vienna, vol. 99, p. 499.

[3] J. A. Serret, *Cours de calcul différentiel et intégral*, vol. 1, 3d ed., Paris, 1886, p. 216. Harnack corrects the error to which Peano refers in the table of corrections of the first volume of Harnack's German translation (1897) of Serret's calculus.

[4] Bertrand, *Calcul différentiel*, Paris, 1864, where the same mistake occurs on p. 504.

[5] I. Todhunter, *A Treatise on the Differential Calculus*, London, 1875, pp. 226–236. G. Battaglini's Italian translation (fifth edition, 1913) of Todhunter's calculus, vol. 1, pp. 255–261, presents the problems under discussion apparently without cognizance of Peano's criteria. The treatment is therefore also unsatisfactory.

[6] J. Pierpont, "Maxima and minima of functions of several variables," *Bulletin of the American Mathematical Society*, 2d series, vol. 4, 1898, pp. 535–539.

2. **Peano's Criterion.**[1] Before taking up the discussion of the Peano surface, it is perhaps well to state the criterion upon which, according to Peano, the existence of maxima and minima depends.

We recall a few preliminary statements concerning definite forms.

A rational homogeneous function of several variables of degree n is called a *form* of the nth degree. In Taylor's expansion for

$$f(x_0 + h, y_0 + k, \cdots)$$

the successive terms are forms of degree 0, 1, 2, $\cdots$ in $h\ k, \cdots$

A form is called *definite*, if it vanishes only when all variables vanish simultaneously. It is *indefinite* if it may assume negative and positive values. Thus, a form of odd degree, which does not vanish identically, is always indefinite. A form may be *neither definite nor indefinite*. This happens for a form which has always the same sign when not zero, but which vanishes for a set $(h, k, \cdots)$ in which *not all* variables vanish. This is what happens in Peano's example.

The criterion for maxima and minima of $f(x, y, \cdots)$ is as follows: *If for* $x = x_0$, $y = y_0$, $\cdots$ *all partial derivatives of less than the nth order vanish, and if in Taylor's expansion for* $z = f(x_0 + h, y_0 + k, \cdots)$, *the term which is a homogeneous function of* $h, k, \cdots$ *of the nth degree is an indefinite form, then* z *is neither a maximum nor a minimum at* $(x_0, y_0, \cdots)$. *If however this form is definite and positive, then* z *is a minimum; if the form is definite and negative, then* z *is a maximum.*

The Peano criterion never gives a wrong result, but it must be understood that the cases where the forms involved are neither definite nor indefinite require individual treatment, like the surface examined hereafter. On the other hand, the other methods to which reference is made actually give wrong results in these cases.

3. **The Peano Surface.** Serret, *l.c.*, p. 219, sets up the criterion: "*the maximum or the minimum takes place when for the values of h and k for which* d^2f *and* d^3f *(third and fourth terms) vanish,* d^4f *(fifth term) has constantly the sign* $-$, *or the sign* $+$." To show that this is erroneous, Peano considers the function

$$z = f(x, y) = (y^2 - 2px)(y^2 - 2qx), \tag{1}$$

in which $p > q > 0$. Putting $x_0 = 0$, $y_0 = 0$ in Taylor's expansion for $f(x_0 + h, y_0 + k)$, there results

$$f(h, k) = 4pqh^2 - 2(p + q)hk^2 + k^4.$$

The second term, d^1f, which is a form of the first degree, here vanishes identically. The system of the terms of the second degree remains different from zero for all values of h and k, except for the values of the sub-set $(0, k)$. Hence the third term d^2f is neither definite nor indefinite. For the set $(0, k)$ the terms of the second and third degree, d^2f and d^3f, vanish and the fifth term is positive. Hence

[1] See *l.c.*, German translation, and the theory on pp. 170–177, preceding maxima and minima. The whole theory of Peano is so simple, and at the same time rigorous, that it should be generally adopted by writers of textbooks.

according to Serret's criterion z would have a minimum, which is false. To show this, let (x, y) approach $(x_0 = 0, y_0 = 0)$ through the function $y^2 = 2lx$. Then

$$z = f(x, \sqrt{2lx}) = 4(l - p)(l - q)x^2,$$

which is negative when $q < l < p$, positive when $l > p$ or $l < q$. Hence there is neither a maximum nor a minimum at $(0, 0)$. Applying Peano's criterion, we note, as above, that the form of the first degree, d^1f, vanishes identically while d^2f remains positive for $h \neq 0$ and vanishes for the subset $(0, k)$. It is neither definite nor indefinite in h and k. The criterion, therefore, does not settle the question in this case. It is by means of the function $y^2 = 2lx$, as shown above (or $x^2 + y^2 = \epsilon^2$, ϵ as small as we please), that the behavior of the function may be investigated.

Geometrically the situation is as follows: The function $z = f(x, y)$ given by (1) is represented by a quartic. For the sake of convenience of construction and modelling we assume it in the form

$$z = -(y^2 - 2px)(y^2 - 2qx), \tag{2}$$

or

$$z = 2(p + q)xy^2 - 4pqx^2 - y^4. \tag{3}$$

Introducing the variable t to make (3) homogeneous ($t = 0$ being the equation of the plane at infinity), we have

$$zt^3 = 2(p + q)xy^2t - 4pqx^2t^2 - y^4. \tag{4}$$

From this is seen that the intersection of the xz-plane with the plane at infinity is a tacnodal line on the quartic. Consequently a plane $y = k$ through this line cuts the quartic in a residual curve which is a parabola, namely,

$$z = 2(p + q)k^2x - 4pqx^2 - k^4.$$

The y-axis has a four-fold contact with the surface. The infinite point of the z-axis is a triple point of the quartic and has the plane at infinity as a three-fold plane. The plane at infinity is a uniplane at the infinite point of the x-axis.

The parabolic cylinder $y^2 = 2lx$ passes through the tacnodal line of the quartic and, consequently, intersects the quartic in a quartic residual curve, whose projection upon the xz-plane has the equation

$$z = -4(l - p)(l - q)x^2,$$

representing a parabola (to be counted twice). On this quartic curve, z is either always negative or always positive (0 for $x = 0$), according to the values of l, as has been found above.

A plane $y = \lambda x$ through the z-axis intersects the surface in a quartic, whose projection upon the xz-plane has the equation

$$z = 2(p + q)\lambda^2x^3 - 4pqx^2 - \lambda^4x^4.$$

Since

$$\frac{dz}{dx} = 6(p + q)\lambda^2 x^2 - 8pqx - 4\lambda^4 x^3,$$

and

$$\frac{d^2z}{dx^2} = 12(p + q)\lambda^2 x - 8pq - 12\lambda^4 x^2,$$

it follows that at (0, 0, 0) for all values of λ the first derivative vanishes and the second derivative has the constant value[1] $-8pq$. Hence:

Every plane through the z-axis, without exception, cuts the surface in a quartic with a maximum at the point (0, 0, 0).

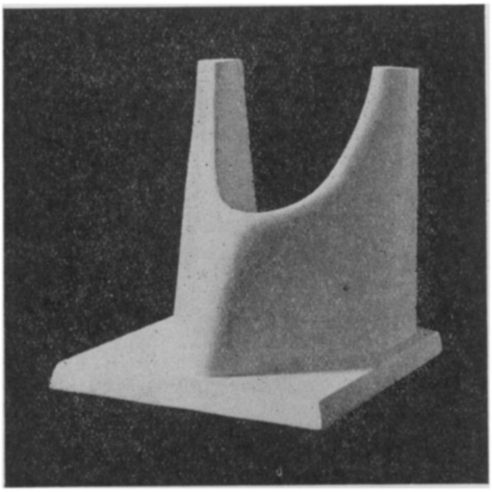

FIG. 1.

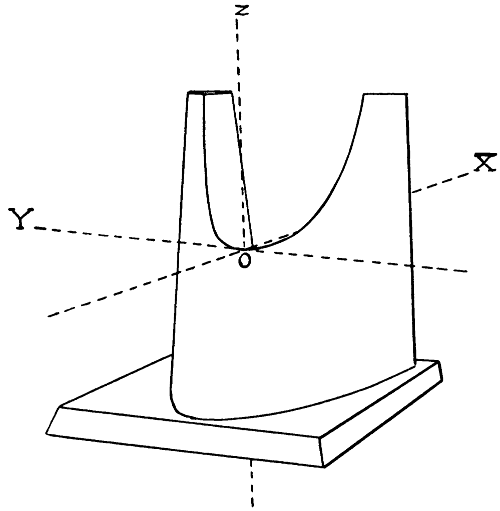

FIG. 2.

The non-critical, purely intuitive mind would ordinarily proclaim such a point a maximum (summit) of the surface. A model of the Peano surface shown in Fig. 1 is a *demonstratio ad oculos* that such is not the case. Fig. 2 shows the location of the surface with respect to the coördinate system. The scale on the z-axis has been chosen as one tenth of that of the x- and y-axes in order to avoid extremely steep slopes. The model represents the true form of the quartic surface defined by the equation

$$z = -\tfrac{1}{10}(y^2 - 5x)(y^2 - x),$$

so that

$$p = 2.5, \qquad q = 0.5.$$

This Peano surface[2] has been constructed on a system of horizontal contour-lines cut out by the pencil of parallel planes $z = \lambda$.

[1] It should be observed, however, that the section for $\lambda = \infty$, that is, the section formed by the yz-plane, cuts the quartic in a curve with a flat point, which projects upon the xz-plane in a segment of a straight line, so that dz^2/dx^2 assumes the indeterminate form $\infty \cdot 0 - 8pq - \infty \cdot 0$. This difficulty disappears for this value of λ if we project the curve of intersection upon the yz-plane. Then for $\lambda = \infty$ the first three derivatives of z with respect to y all vanish, while the fourth derivative is negative. Under these circumstances the flat point may still be defined as a maximum.

[2] The mould for this plaster model has been preserved, so that casts might be made for parties wishing to possess such a model in their collection.